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BHATTACHARYA AND HOLLA DISTRIBUTION AND SOME OF ITS INTERESTING PROPERTIES AND APPLICATIONS

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ABSTRACT

In this paper we consider Bhattacharya and Holla's (1965) distribution and derive its important structural properties which have not been studied so far. The distribution has very interesting properties when transformed in trigonometric functions which are also useful and alternative expression for obtaining its higher order moments. We also make an attempt to obtain estimate of its parameters. The model is fitted to the set of observed data and relative precision and a comparison has also been made.

Key words: Generating function, trigonometric function, compound distribution.

1. Introduction

Gurland (1957) studied interrelations among compound and generalized distributions. Gurland (1958) also defined a general class of contagious distributions. Patil (1964) studied on certain compound Poisson and compound binomial distribution. Blischke (1963) defined mixture of discrete distributions. Cohen (1963) obtained estimation in mixtures of discrete distributions. Johnson and Kotz (1969) and Johnson, Kotz and Kemp (1992) also discussed some mixture, compound and contagious distributions.

One of the interesting and useful compound distributions is defined by Bhattacharya and Holla (1965). This distribution showed its importance in the theory of accident proneness. This distribution is obtained by mixing Poisson distribution with rectangular distribution denoted as Poisson $\hat{\theta}$ rectangular (a, b) as

$$P(X=k) \quad = \frac{1}{b-a}\int_a^b \frac{e^{-\theta}\theta^k}{k!}d\theta\,; \quad k = 0, 1,2,\ldots \tag{1.1}$$

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$$P(X=k)=\frac{1}{b-a}\left[e^{-a}\left(1+\frac{a}{1!}+\frac{a^2}{2!}+\ldots+\frac{a^k}{k!}\right)-e^{-b}\left(1+\frac{b}{1!}+\frac{b^2}{2!}+\ldots+\frac{b^k}{k!}\right)\right] \quad (1.2)$$

$$k=0,\ 1,2,\ldots$$

This is known as Bhattacharya and Holla (1965) distribution.

As regards the applications, the use of (1.1) in the field of the Poisson queuing system, accident statistics and acceptance sampling plans of manufactured articles based on the counting of defects, etc. are well known, the variations in service facilities from one server to another in a parallel service channels may result in fluctuations in individuals in the first case, in variations in the accident liabilities from individual to individual in the second case and, in the later case, in an inevitable and continuous changing in manufacturing conditions leading to fluctuations in the Poisson distribution parameter involved.

In this study we derive its important structural properties which have not been done earlier so far. The important feature of this paper is that to obtain moments and moment generating function in terms of trigonometric functions. This transformed form gives very interesting properties of the distribution and higher order moments can also be obtained using this transformed form. This expression is an alternative for obtaining moments of the proposed model. We also provide estimation of the parameters of the distribution. The Poisson model has been fitted by many statisticians. The same data has been fitted to this model. This model gives better result as compared to Poisson distribution.

2. Properties of Compound Distribution

2.1. Moments about origin

The first four moments of (1.1) about origin are derived as

$$\mu_1' = \frac{b+a}{2} \quad (2.1)$$

$$\mu_2' = \frac{a^2+b^2+ab}{3}+\frac{b+a}{2} \quad (2.2)$$

$$\mu_3' = \frac{b^3+ba^2+ab^2+a^3}{4}+(a^2+b^2+ab)+\frac{b+a}{2} \quad (2.3)$$

$$\mu_4' = \frac{a^4 + b^4 + a^3 b + ab^3 + a^2 b^2}{5} + \frac{3(a^3 + b^3 + a^2 b + ab^2)}{2} + \frac{7(a^2 + b^2 + ab)}{3} + \frac{a+b}{2} \tag{2.4}$$

The central moments of (1.1) are obtained as

$$\mu_2 = \frac{(b-a)^2}{12} + \frac{b+a}{2} \tag{2.5}$$

$$\mu_3 = \frac{b+a}{2} + \frac{(b-a)^2}{4} \tag{2.6}$$

$$\mu_4 = \frac{(b-a)^4}{80} + \frac{(a+b)(b-a)^2}{4} + \frac{4a^2 + 4b^2 + ab}{3} + \frac{a+b}{2} \tag{2.7}$$

2.2.Moment Generating Function

We also find the moment generating function of (1.1)

$$M_x(t) = \frac{e^{-a(1-e^t)} - e^{-b(1-e^t)}}{(b-a)(1-e^t)}$$

$$M_x(t) = \frac{1}{b-a}\left[(b-a) - \frac{(b^2 - a^2)}{2!}(1-e^t) + \frac{(b^3 - a^3)}{3!}(1-e^t)^2 - \frac{(b^4 - a^4)}{4!}(1-e^t)^3 + \ldots \right] \tag{2.8}$$

2.3.Characteristic Function

$$\phi_X(t) = \frac{e^{-a(1-e^{it})} - e^{-b(1-e^{it})}}{(b-a)(1-e^{it})}$$

$$\phi_x(t) = \frac{1}{b-a}\left[(b-a) - \frac{(b^2 - a^2)}{2!}(1-e^{it}) + \frac{(b^3 - a^3)}{3!}(1-e^{it})^2 - \frac{(b^4 - a^4)}{4!}(1-e^{it})^3 + \ldots \right] \tag{2.9}$$

2.4. Cumulant Generating Function

It provides a technique for obtaining moments of the distribution in a very simple and convenient way. The provided Log M_x (t) is a convergent function of t.

$$K_X(t)=\left\{\left[\frac{b^2-a^2}{2!(b-a)}\left(t+\frac{t^2}{2!}+\ldots\right)+\frac{b^3-a^3}{3!(b-a)}\left(t+\frac{t^2}{2!}+\ldots\right)^2\right.\right.$$

$$\left.+\frac{b^4-a^4}{4!(b-a)}\left(t+\frac{t^2}{2!}+\ldots\right)^3+\cdots\right]$$

$$-\frac{1}{2}\left[\frac{b^2-a^2}{2!(b-a)}\left(t+\frac{t^2}{2!}+\ldots\right)+\frac{b^3-a^3}{3!(b-a)}\left(t+\frac{t^2}{2!}+\ldots\right)^2\right.$$

$$\left.+\frac{b^4-a^4}{4!(b-a)}\left(t+\frac{t^2}{2!}+\ldots\right)^3+\cdots\right]^2$$

$$+\frac{1}{3}\left[\frac{b^2-a^2}{2!(b-a)}\left(t+\frac{t^2}{2!}+\ldots\right)+\frac{b^3-a^3}{3!(b-a)}\left(t+\frac{t^2}{2!}+\ldots\right)^2\right.$$

$$\left.\left.+\frac{b^4-a^4}{4!(b-a)}\left(t+\frac{t^2}{2!}+\ldots\right)^3+\cdots\right]^3-\cdots\right\} \tag{2.10}$$

$$K_X(t)=Y-\frac{1}{2}Y^2+\frac{1}{3}Y^3-\frac{1}{4}Y^4+\ldots$$

where

$$Y=\frac{b^2-a^2}{2!(b-a)}\left(t+\frac{t^2}{2!}+\ldots\right)+\frac{b^3-a^3}{3!(b-a)}\left(t+\frac{t^2}{2!}+\ldots\right)^2+\frac{b^4-a^4}{4!(b-a)}\left(t+\frac{t^2}{2!}+\ldots\right)^3+\cdots$$

Now equating the coefficients of like powers of t on both sides of equation (2.10), we get the cumulants as

$$K_1=\frac{a+b}{2} \tag{2.11}$$

$$K_2 = \frac{(b-a)^2}{12} + \frac{b+a}{2} \tag{2.12}$$

Similarly

$$K_3 = \frac{b+a}{2} + \frac{(b-a)^2}{4} \tag{2.13}$$

$$\mathrm{K}_4 = -\frac{(\mathrm{b}-\mathrm{a})^4}{120} + \frac{7(\mathrm{b}-\mathrm{a})^2}{12} - \frac{\mathrm{a}+\mathrm{b}}{2} \tag{2.14}$$

$$K_4 + 3K_2^2 = \frac{(b-a)^4}{80} + \frac{(a+b)(b-a)^2}{4} + \frac{4a^2+4b^2+ab}{3} + \frac{a+b}{2} \tag{2.15}$$

The (2.12), (2.13) and (2.15) are also central moments equal to (2.5) to (2.7)

2.5.Skewness

The Skewness of the (1.1) can be obtained by using the expressions given in (2.5) and (2.6) as

$$\gamma_1 = \sqrt{108}\,\frac{\left[2(\mathrm{b}+\mathrm{a}) + (\mathrm{b}-\mathrm{a})^2\right]}{\left[6(\mathrm{b}+\mathrm{a}) + (\mathrm{b}-\mathrm{a})^2\right]^{3/2}} \tag{2.16}$$

It is clear from above for given a and b the Skewness is positive for $\frac{(\mathrm{b}-\mathrm{a})^2}{\mathrm{b}+\mathrm{a}} > -2$, and the Skewness is negative for $-6 < \frac{(\mathrm{b}-\mathrm{a})^2}{\mathrm{b}+\mathrm{a}} < -2$ provided $\mathrm{b} + \mathrm{a} \neq 0$. In case $\mathrm{b} + \mathrm{a} = 0$

$$\gamma_1 = \frac{\sqrt{108}}{\mathrm{b}-\mathrm{a}}$$

This means Skewness is inversely proportional to (b – a)

2.6.Kurtosis

$$\beta_2 = 3 + \frac{6}{5}\,\frac{\left[60(\mathrm{a}+\mathrm{b}) + 70(\mathrm{b}-\mathrm{a})^2 - (\mathrm{b}-\mathrm{a})^4\right]}{\left[36(\mathrm{a}+\mathrm{b})^2 + 12(\mathrm{b}+\mathrm{a})(\mathrm{b}-\mathrm{a})^2 + (\mathrm{b}-\mathrm{a})^4\right]} \tag{2.17}$$

For given values of a and b the Kurtosis is greater than 3 if the condition $\frac{60}{(b-a)^2-70} > \frac{(b-a)^2}{a+b}$ provided a + b ≠ 0 is satisfied, this reveals the models (1.1) and (1.2) are leptokurtic. However, if the condition $\frac{60}{(b-a)^2-70} < \frac{(b-a)^2}{a+b}$ provided a + b ≠ 0 is satisfied. The models are platykurtic. In case a + b = 0, $\beta_2 = 3 + \frac{84}{(b-a)^2} - 6/5$

This means coefficient of Kurtosis is a decreasing function of (b – a). The models take normal form at b – a =$\pm\sqrt{70}$.

3. Recurrence Relation for Moment Generating Function and Moments in terms of Trigonometric function

We have MGF (2.8) which can also be written as

$$M_X(t) = \frac{1}{b-a}\sum_{r=0}^{\infty}\frac{(b^{r+1}-a^{r+1})}{(r+1)!}(-1)^r(1-e^t)^r \tag{3.1}$$

Taking the n-th term (3.1) and let

$$D = \frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)!}(-1)^n(1-e^t)^n \tag{3.2}$$

$$= A_n(1-e^t)^n \tag{3.3}$$

where $A_n = \frac{(b^{n+1}-a^{n+1})}{(b-a)(n+1)}(-1)^n$

$$\text{Let } D = A_n(1-e^t)^n \tag{3.4}$$

Put $e^t = \sin^2\theta$

Since we find moments at t = 0, when t = 0, θ = π/2 so in the transformed case we will find moments of the distribution for θ = π/2

$$D = A_n \cos^{2n}\theta \tag{3.5}$$

where $e^t = \sin^2\theta$

$$\frac{dD}{dt} = D' = A_n\left[\frac{d}{d\theta}\cos^{2n}\theta\right]\frac{d\theta}{dt}$$

$$D^1 = -A_n \; n \, Cos^{2n-2}\theta \; Sin^2 \theta \tag{3.6}$$

$$D^2 = D' + n(n-1) \, A_n Cos^{2n-4}\theta \, Sin^4\theta \tag{3.7}$$

$$D^3 = D^2 + n(n-1)A_n \big[4 \, Cos^{2n-4}\theta \, Sin^3\theta \, Cos \, \theta$$

$$- (2n-4)Cos^{2n-5} Sin \, \theta \, Sin^4\theta\big]\frac{Sin\,\theta}{2 \, \mathrm{Cos} \, \theta}$$

$$D^3 = -2 \, D' + 3 \, D^2 - n(n-1)(n-2)A_n \; Cos^{2n-6}\theta \, Sin^6\theta \tag{3.8}$$

$$D^4 = 6D^1 - 11D^2 + 6D^3 + n(n-1)(n-2)(n-3)A_n \; Cos^{2n-8}\theta \, Sin^8\theta \tag{3.9}$$

$$D^5 = -24D^1 + 50D^2 - 35D^3 + 10D^4$$

$$- n(n-1)(n-2)(n-3)(n-4)A_n \; Cos^{2n-10}\theta \, Sin^{10}\theta \tag{3.10}$$

$$\begin{aligned} D^6 &= 120D^1 - 274D^2 + 225D^3 - 85D^4 + 15D^5 \\ &+ n(n-1)\ldots(n-5)A_n \; Cos^{2n-12}\theta \; Sin^{12}\theta \end{aligned} \tag{3.11}$$

$$D^7 = -720D^1 + 1764D^4 - 1624D^3 + 735D^4 - 175D^5 + 21D^6$$

$$- n(n-1)\ldots(n-6)A_n \; Cos^{2n-14}\theta \, Sin^{14}\theta \tag{3.12}$$

Where D^1, D^2, D^3, D^4 … are 1^{st}, 2^{nd}, 3^{rd}, 4^{th} … derivatives of D (3.4)
Now writing the general term as

$$D^n = (-1)^n \big\{ \big[a_1 D^1 - a_2 D^2 + a_3 D^3 - a_4 D^4 + \ldots - (n-1)\text{terms}\big] + n! \, A_n Sin^{2n}\theta \big\} \tag{3.15}$$

$$D^n = (-1)^n \big\{ f_{n-1}(D) + n! \, A_n Sin^{2n}\theta \big\} \tag{3.16}$$

where n = 2, 3, 4, …
where $f_{n-1}(D) = a_1 D^1 - a_2 D^2 + a_3 D^3 - a_4 D^4 + \ldots$, where n = 2, 3, 4, …
where the coefficient a_1, a_2, a_3, … a $_{n-1}$ are determined by using the following iterative table which is discussed below:

$f_n(D)$	n	a_1	a_2	a_3	a_4	a_5	a_6
$f_1(D)$	1	1					
$f_2(D)$	2	2	3				
$f_3(D)$	3	6	11	6			
$f_4(D)$	4	24	50	35	10		
$f_5(D)$	5	120	274	225	85	15	
$f_6(D)$	6	720	1764	1624	735	175	21

The coefficient matrix is formed by a straightforward method. First column, i.e. coefficient column a_1 is obtained by writing down n! in each row for each n = 1, 2, …. Diagonal of the matrix, i.e. coefficient of a_1, a_2, a_3, … along the diagonal is obtained by summing up the integers up to n = 1, 2, 3, … in each row. In other words, we use $n(n+1)/2$ for writing down diagonal elements.

If c_{ij} denotes the element of the given matrix being in the i-th row and j-th column then c_{ij} $c_{ij} = i\, c_{(i-1)j} + c_{(i-1)(j-1)}$ for every i=2,…,n , j=2,3,…, i-1

Moreover, the above recurrence relation can also be written in matrix form as:

$$\begin{bmatrix} D^2 \\ -D^3 \\ D^4 \\ -D^5 \\ D^6 \\ -D^7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 & 0 \\ 6 & -11 & 6 & 0 & 0 & 0 \\ 24 & -50 & 35 & -10 & 0 & 0 \\ 120 & -274 & 225 & -85 & 15 & 0 \\ 720 & -1764 & 1624 & -735 & 175 & -21 \end{bmatrix} \begin{bmatrix} D^1 \\ D^2 \\ D^3 \\ D^4 \\ D^5 \\ D^6 \end{bmatrix} + \begin{bmatrix} 2\,A_2 \mathrm{Sin}^4\theta \\ 6\,A_3 \mathrm{Sin}^6\theta \\ 24\,A_4 \mathrm{Sin}^8\theta \\ 120\,A_5 \mathrm{Sin}^{10}\theta \\ 720\,A_6 \mathrm{Sin}^{12}\theta \\ 5040\,A_7 \mathrm{Sin}^{14}\theta \end{bmatrix}$$

Solving this matrix equation we can obtain moments of any order. (3.17)

4. Estimation

We estimate a and b by moments method, $m_r = \mu'_r$ where μ'_r and m_r are population moments and sample moments about origin respectively. The estimates of a and b are

$$\hat{b} = \bar{x} + \sqrt{3(s^2 - \bar{x})} \tag{4.1}$$

$$\hat{a} = \bar{x} - \sqrt{3(s^2 - \bar{x})} \tag{4.2}$$

where $\bar{x}$ and s^2 are sample mean and variance.

5. Goodness of Fit

The proposed model seems to explain observed data with great accuracy. The model is fitted with the observed data in comparison with Poisson distribution. It seems that proposed model is relatively more appropriate for natural behaviour.

We present the fitting to a data to which Poisson has been be fitted by Consul and Jain (1973). The same set of observed data is fitted to the proposed model (1.1) which has not been fitted earlier. The relative precision is also presented graphically in the following tables given below.

Table 5.1. Accidents of 647 women working on H.E. Shells during five weeks

No. of Accidents	Observed Frequency	Expected frequencies	
		Poisson Distribution PD	Bhattacharya and Holla Distribution BHD
0	447	406	454
1	132	189	113
2	42	44	59
3	21	7	16
4	3	1	4
5	2	0	1
TOTAL	647	647	647
Mean	0.465224		
Variance	0.690826		
Estimates		λ =0.465224	a= -0.356
			b= 1.287
χ^2		49.42	11.008
d.f		2	1

Table 5.2.

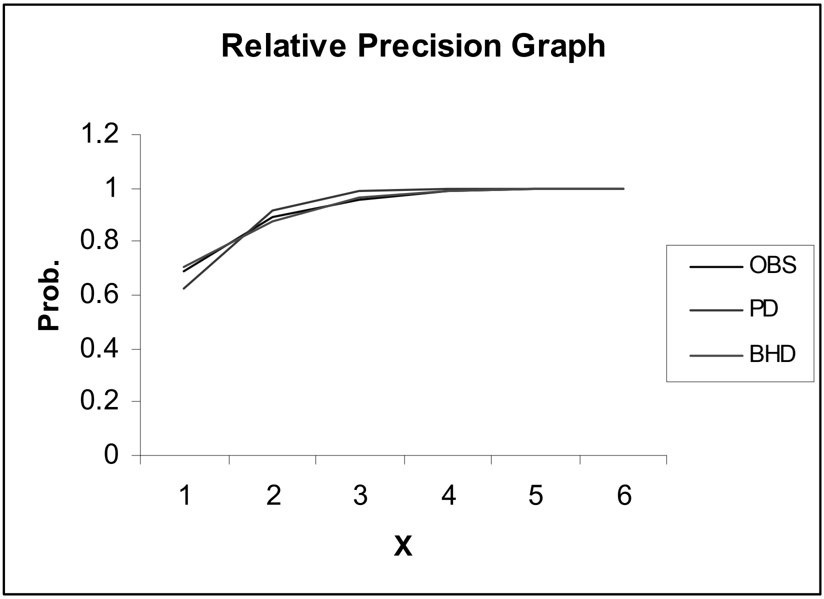
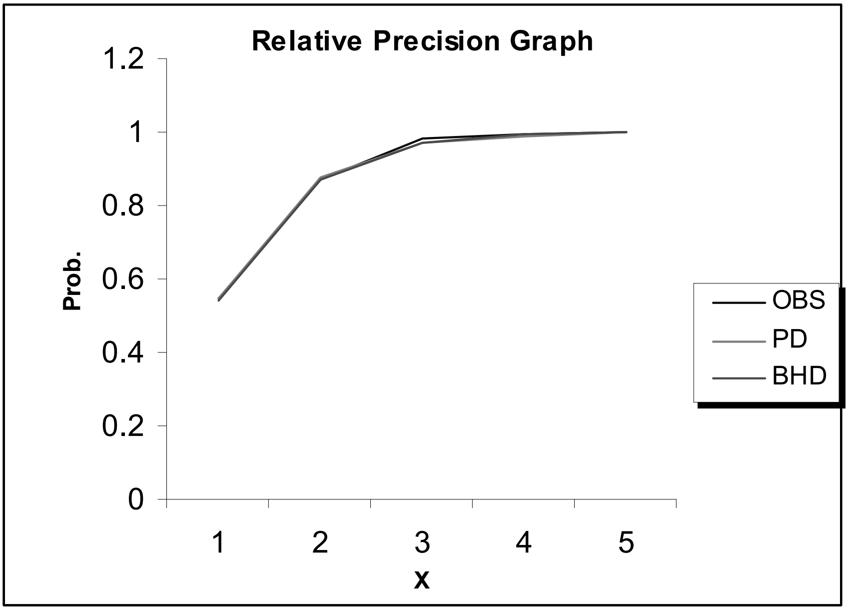


Table 5.3. Death due to horse kicks in the Prussian army

Number of deaths	Observed Frequency	Expected frequencies	
		Poisson Distribution PD	Bhattacharya and Holla Distribution BHD
0	109	109	108
1	65	66	66
2	22	19	20
3	3	4	5
4	1	2	1
TOTAL	200	200	200
Mean	0.61000		
Variance	0.610955		
Estimates		λ= 0.61000	a=0.5565
			b= 0.6635
χ^2		1.24	1.006
d.f		2	1

Table 5.4.



It is evident from the tables 5.1 , 5.2, 5.3 and 5.4 that the values of chi-square, and graph of relative precision in all the cases, the model gives the remarkable best fit as compared to Poisson distributions.

Conclusion

In this paper, we considered the Bhattacharya and Holla (1965) distribution as possible alternative to accident statistics and acceptance sampling plans of manufactured articles based on the counting of defects and for analyzing data from different behavior. We studied various properties of the model interims of generating function (MGF). The model has interesting properties when moment generating function is transformed in trigonometric functions.

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